

# CATEGORICAL ENTROPY FOR FOURIER-MUKAI TRANSFORMS ON GENERIC ABELIAN SURFACES.

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ABSTRACT. In this note, we shall compute the categorical entropy of an autoequivalence on a generic abelian surface.

## 0. INTRODUCTION

In [1], Dimitrov, Haiden, Katzarkov, and Kontsevich introduced a categorical entropy  $h_t(\Phi)$  ( $t \in \mathbb{R}$ ) for an endofunctor  $\Phi$  of a triangulated category with a split generator. For endofunctors of the derived category  $\mathbf{D}(X)$  of coherent sheaves on a smooth projective variety  $X$ , Kikuta and Takahashi [3], [4] studied the entropy. In particular they proved that  $h_0$  coincides with the topological entropy for an endofunctor  $\mathbf{L}f^*$  induced by a surjective endomorphism  $f$  of smooth projective variety [4, Thm. 5.4] and an autoequivalence of  $\mathbf{D}(X)$  if  $\dim X = 1$  [3] or  $\pm K_X$  is ample [4, Thm. 5.6]. They also conjectured that a Gromov-Yomdin type result holds, that is,  $h_0(\Phi) = \log \rho(\Phi)$  ([4, Conjecture 5.3]) where  $\rho(\Phi)$  is the spectral radius of the action of  $\Phi$  on the algebraic cohomology group  $H^*(X, \mathbb{Q})_{\text{alg}}$ . It seems that there are only a few example of computation of categorical entropy, and it may be interesting to add more examples. In this note, we shall give an almost trivial example of the computation. Thus we shall compute the entropy for special autoequivalences on abelian surfaces. For an abelian variety, Orlov [9] proved that the kernel of an equivalence is a sheaf up to shift. So we can expect that the entropy which measures the complexity of an equivalence is simple. For a special equivalence on an abelian surface, we shall check that our expectation is true.

To be more precise, let  $X$  be an abelian surface and  $H$  an ample divisor. We set  $L := \mathbb{Z} \oplus \mathbb{Z}H \oplus \mathbb{Z}\varrho_X$ , where  $\varrho_X$  is the fundamental class of  $X$ . Let  $\Phi : \mathbf{D}(X) \rightarrow \mathbf{D}(X)$  be a Fourier-Mukai functor which preserves  $L$ . In Theorem 2.2, we shall compute  $h_t(\Phi)$ . Combining a recent paper of Ikeda [2], we also check that the conjecture of Kikuta and Takahashi holds for equivalences on abelian surfaces (Proposition 2.10).

We also remark that there is a symplectic manifold of generalized Kummer type which has an automorphism of infinite order. This is an analogue of a recent result of Ouchi [11]. By the absence of spherical objects, our example is almost trivial.

## 1. FOURIER-MUKAI TRANSFORMS ON AN ABELIAN SURFACE

**1.1. Notation.** We denote the category of coherent sheaves on  $X$  by  $\text{Coh}(X)$  and the bounded derived category of  $\text{Coh}(X)$  by  $\mathbf{D}(X)$ . A Mukai lattice of  $X$  consists of  $H^{2*}(X, \mathbb{Z}) := \bigoplus_{i=0}^2 H^{2i}(X, \mathbb{Z})$  and an integral bilinear form  $\langle \ , \ \rangle$  on  $H^{2*}(X, \mathbb{Z})$ :

$$\langle x_0 + x_1 + x_2\varrho_X, y_0 + y_1 + y_2\varrho_X \rangle := (x_1, y_1) - x_0y_2 - x_2y_0 \in \mathbb{Z},$$

where  $x_1, y_1 \in H^2(X, \mathbb{Z})$ ,  $x_0, x_2, y_0, y_2 \in \mathbb{Z}$  and  $\varrho_X \in H^4(X, \mathbb{Z})$  is the fundamental class of  $X$ . We also introduce the algebraic Mukai lattice as the pair of  $H^*(X, \mathbb{Z})_{\text{alg}} := \mathbb{Z} \oplus \text{NS}(X) \oplus \mathbb{Z}$  and  $\langle \ , \ \rangle$  on  $H^*(X, \mathbb{Z})_{\text{alg}}$ . For  $x = x_0 + x_1 + x_2\varrho_X$  with  $x_0, x_2 \in \mathbb{Z}$  and  $x_1 \in H^2(X, \mathbb{Z})$ , we also write  $x = (x_0, x_1, x_2)$ . For  $E \in \mathbf{D}(X)$ ,  $v(E) := \text{ch}(E)$  denotes the Mukai vector of  $E$ .

For  $\mathbf{E} \in \mathbf{D}(X \times Y)$ , we set

$$\Phi_{X \rightarrow Y}^{\mathbf{E}}(x) := \mathbf{R}p_{Y*}(\mathbf{E} \otimes p_X^*(x)), \ x \in \mathbf{D}(X),$$

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where  $p_X, p_Y$  are projections from  $X \times Y$  to  $X$  and  $Y$  respectively. Let  $\text{Eq}(\mathbf{D}(X), \mathbf{D}(Y))$  be the set of equivalences between  $\mathbf{D}(X)$  and  $\mathbf{D}(Y)$ . We set

$$\begin{aligned} & \text{Eq}_0(\mathbf{D}(Y), \mathbf{D}(Z)) \\ & := \left\{ \Phi_{Y \rightarrow Z}^{\mathbf{E}[2k]} \in \text{Eq}(\mathbf{D}(Y), \mathbf{D}(Z)) \mid \mathbf{E} \in \text{Coh}(Y \times Z), k \in \mathbb{Z} \right\}, \\ & \mathcal{E}(Z) := \bigcup_Y \text{Eq}_0(\mathbf{D}(Y), \mathbf{D}(Z)), \\ & \mathcal{E} := \bigcup_Z \mathcal{E}(Z) = \bigcup_{Y, Z} \text{Eq}_0(\mathbf{D}(Y), \mathbf{D}(Z)). \end{aligned}$$

Note that  $\mathcal{E}$  is a groupoid with respect to the composition of the equivalences.

For an object  $E \in \mathbf{D}(X)$  with  $\text{rk } E \neq 0$ , we set  $\mu(E) := c_1(E)/\text{rk } E$ .

**1.2. Semi-homogeneous sheaves.** We collect some properties of semi-homogeneous sheaves on an abelian surface [6].

**Proposition 1.1.** *For a coherent sheaf  $E$  on  $X$ , the following conditions are equivalent.*

- (i)  $E$  is a semi-homogeneous sheaf.
- (ii)  $E$  is a semi-stable sheaf with  $\langle v(E)^2 \rangle = 0$  with respect to an ample divisor  $H$ .

**Proposition 1.2** (cf. [9]). *Let  $\Phi : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$  be an equivalence. For a semi-homogeneous sheaf  $E$  on  $X$ , there is an integer  $n$  such that  $\Phi(E)[n]$  is a semi-homogeneous sheaf.*

**Proposition 1.3** ([14, Prop. 4]). *Let  $E$  and  $F$  be semi-homogeneous sheaves.*

- (i) Assume that  $E$  and  $F$  are locally free sheaves.
    - (a) If  $\langle v(E), v(F) \rangle > 0$ , then  $\text{Hom}(E, F) = \text{Ext}^2(E, F) = 0$ .
    - (b) If  $\langle v(E), v(F) \rangle < 0$ , then  $(\mu(E), H) \neq (\mu(F), H)$ ,  $\text{Ext}^1(E, F) = 0$  and
- $$(1.1) \quad \begin{cases} \text{Hom}(E, F) = 0, & (\mu(E), H) > (\mu(F), H) \\ \text{Ext}^2(E, F) = 0, & (\mu(F), H) > (\mu(E), H). \end{cases}$$
- (ii) Assume that  $E$  is locally free and  $F$  is a torsion sheaf.
    - (a) If  $\langle v(E), v(F) \rangle > 0$ , then  $\text{Hom}(E, F) = \text{Ext}^2(E, F) = 0$ .
    - (b) If  $\langle v(E), v(F) \rangle < 0$ , then  $\text{Ext}^1(E, F) = \text{Ext}^2(E, F) = 0$ .
  - (iii) Assume that  $E$  and  $F$  are torsion sheaves. Then  $\langle v(E), v(F) \rangle \geq 0$ . If  $\langle v(E), v(F) \rangle > 0$ , then  $\text{Hom}(E, F) = \text{Ext}^2(E, F) = 0$ .

*Remark 1.4.* If  $(D^2) > 0$ , then  $D$  is ample if and only if  $(D, H_0) > 0$  for an ample divisor  $H_0$ . Since  $-\langle v(E), v(F) \rangle = \text{rk } E \text{ rk } F ((\mu(E) - \mu(F))^2)/2$ ,  $\langle v(E), v(F) \rangle < 0$  implies  $\mu(E) - \mu(F)$  is ample or  $\mu(F) - \mu(E)$  is ample.

**1.3. Cohomological Fourier-Mukai transforms.** We collect some results on the Fourier-Mukai transforms on abelian surfaces  $X$  with  $\text{rk NS}(X) = 1$ . Let  $H_X$  be the ample generator of  $\text{NS}(X)$ . We shall describe the action of Fourier-Mukai transforms on the cohomology lattices in [12]. For  $Y \in \text{FM}(X)$ , we have  $(H_Y^2) = (H_X^2)$ . We set  $D := (H_X^2)/2$ . In [12, sect. 6.4], we constructed an isomorphism of lattices

$$\begin{aligned} \iota_X : (H^*(X, \mathbb{Z})_{\text{alg}}, \langle \cdot, \cdot \rangle) & \xrightarrow{\sim} (\text{Sym}_2(\mathbb{Z}, D), B), \\ (r, dH_X, a) & \mapsto \begin{pmatrix} r & d\sqrt{D} \\ d\sqrt{D} & a \end{pmatrix}, \end{aligned}$$

where  $\text{Sym}_2(\mathbb{Z}, D)$  is given by

$$\text{Sym}_2(\mathbb{Z}, D) := \left\{ \begin{pmatrix} x & y\sqrt{D} \\ y\sqrt{D} & z \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\},$$

and the bilinear form  $B$  on  $\text{Sym}_2(\mathbb{Z}, D)$  is given by

$$B(X_1, X_2) := 2Dy_1y_2 - (x_1z_2 + z_1x_2)$$

for  $X_i = \begin{pmatrix} x_i & y_i\sqrt{D} \\ y_i\sqrt{D} & z_i \end{pmatrix} \in \text{Sym}_2(\mathbb{Z}, D)$  ( $i = 1, 2$ ).

Each  $\Phi_{X \rightarrow Y} \in \text{Eq}_0(\mathbf{D}(X), \mathbf{D}(Y))$  gives an isometry

$$(1.2) \quad \iota_Y \circ \Phi_{X \rightarrow Y}^H \circ \iota_X^{-1} \in \text{O}(\text{Sym}_2(\mathbb{Z}, D)),$$

where  $\text{O}(\text{Sym}_2(\mathbb{Z}, D))$  is the isometry group of the lattice  $(\text{Sym}_2(\mathbb{Z}, D), B)$ . Thus we have a map

$$\eta : \mathcal{E} \rightarrow \text{O}(\text{Sym}_2(\mathbb{Z}, D))$$

which preserves the structures of multiplications.

**Definition 1.5.** We set

$$\widehat{G} := \left\{ \begin{pmatrix} a\sqrt{r} & b\sqrt{s} \\ c\sqrt{s} & d\sqrt{r} \end{pmatrix} \middle| \begin{array}{l} a, b, c, d, r, s \in \mathbb{Z}, r, s > 0 \\ rs = D, adr - bcs = \pm 1 \end{array} \right\},$$

$$G := \widehat{G} \cap \mathrm{SL}(2, \mathbb{R}).$$

We have an action  $\cdot$  of  $\widehat{G}$  on the lattice  $(\mathrm{Sym}_2(\mathbb{Z}, D), B)$ :

$$(1.3) \quad g \cdot \begin{pmatrix} r & d\sqrt{D} \\ d\sqrt{D} & a \end{pmatrix} := g \begin{pmatrix} r & d\sqrt{D} \\ d\sqrt{D} & a \end{pmatrix} {}^t g, \quad g \in \widehat{G}.$$

Thus we have a homomorphism:

$$\alpha : \widehat{G}/\{\pm 1\} \rightarrow \mathrm{O}(\mathrm{Sym}_2(\mathbb{Z}, D)).$$

**Theorem 1.6** ([12, Thm. 6.16, Prop. 6.19]). *Let  $\Phi \in \mathrm{Eq}_0(\mathbf{D}(Y), \mathbf{D}(X))$  be an equivalence.*

- (1)  $v_1 := v(\Phi(\mathcal{O}_Y))$  and  $v_2 := \Phi(\varrho_Y)$  are positive isotropic Mukai vectors with  $\langle v_1, v_2 \rangle = -1$  and we can write

$$\begin{aligned} v_1 &= (p_1^2 r_1, p_1 q_1 H_X, q_1^2 r_2), \quad v_2 = (p_2^2 r_2, p_2 q_2 H_X, q_2^2 r_1), \\ p_1, q_1, p_2, q_2, r_1, r_2 &\in \mathbb{Z}, \quad p_1, r_1, r_2 > 0, \\ r_1 r_2 &= D, \quad p_1 q_2 r_1 - p_2 q_1 r_2 = 1. \end{aligned}$$

- (2) We set

$${}^t\theta(\Phi) := \pm \begin{pmatrix} p_1 \sqrt{r_1} & p_2 \sqrt{r_2} \\ q_1 \sqrt{r_2} & q_2 \sqrt{r_1} \end{pmatrix} \in G/\{\pm 1\}.$$

Then  ${}^t\theta(\Phi)$  is uniquely determined by  $\Phi$  and we have a map

$${}^t\theta : \mathcal{E} \rightarrow G/\{\pm 1\}.$$

- (3) The action of  ${}^t\theta(\Phi)$  on  $\mathrm{Sym}_2(\mathbb{Z}, D)$  is the action of  $\Phi$  on the algebraic Mukai lattice:

$$\iota_X \circ \Phi(v) = {}^t\theta(\Phi) \cdot \iota_Y(v).$$

Thus we have the following commutative diagram:

$$(1.4) \quad \begin{array}{ccc} \mathcal{E} & & \\ {}^t\theta \downarrow & \searrow \eta & \\ \widehat{G}/\{\pm 1\} & \xrightarrow{\alpha} & \mathrm{O}(\mathrm{Sym}_2(\mathbb{Z}, D)) \end{array}$$

For  $\Phi : \mathrm{Eq}_0(\mathbf{D}(X), \mathbf{D}(X))$  preserving the sublattice  $L := \mathbb{Z} \oplus \mathbb{Z}H_X \oplus \mathbb{Z}\varrho_X$ , it is easy to see that the same proof of Theorem 1.6 works. Thus replacing  $H^*(X, \mathbb{Z})_{\mathrm{alg}}$  by  $L$ , we have a similar claims to Theorem 1.6. From now on, we identify the Mukai lattice  $L$  with  $\mathrm{Sym}_2(\mathbb{Z}, D)$  via  $\iota_X$ . Then for  $g \in \widehat{G}$  and  $v \in L$ ,  $g \cdot v$  means  $\iota_X(g \cdot v) = g \cdot \iota_X(v)$ . We also set  $H := H_X$ . By [15, Lem. 2.5], we have the following.

**Lemma 1.7.** *For*

$$(1.5) \quad A = \begin{pmatrix} a & b\sqrt{D} \\ c\sqrt{D} & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}, ad - bcD = 1,$$

there is a Fourier-Mukai transform  $\Phi_{X \rightarrow X}^{\mathbf{E}_A} : \mathbf{D}(X) \rightarrow \mathbf{D}(X)$  such that  $\Phi_{X \rightarrow X}^{\mathbf{E}_A}(L) = L$  with  ${}^t\theta(\Phi_{X \rightarrow X}^{\mathbf{E}_A}) = A$ .

Replacing  $A$  by  $-A$  if necessary, we assume that  $\mathrm{tr} A \geq 0$ .

**Definition 1.8.** For a matrix  $A$  in (1.5) such that  $\mathrm{tr} A \geq 0$ , let  $\Phi_{X \rightarrow X}^{\mathbf{E}_A} : \mathbf{D}(X) \rightarrow \mathbf{D}(X)$  such that  $\mathbf{E}_A \in \mathrm{Coh}(X \times X)$  and  $\Phi_{X \rightarrow X}^{\mathbf{E}_A}(L) = L$  with  ${}^t\theta(\Phi_{X \rightarrow X}^{\mathbf{E}_A}) = A$ .

We note that

$$(1.6) \quad v((\mathbf{E}_A)_{| \{x\} \times X}) = (b^2 D, bdH, d^2), \quad v((\mathbf{E}_A^\vee)_{| X \times \{x\}}) = (b^2 D, -baH, a^2).$$

*Remark 1.9.* If  $\mathbf{E}'_A$  also induces the same cohomological transform  $\Phi$ , then there is an isomorphism  $f : X \rightarrow X$  and a line bundle  $P \in \mathrm{Pic}^0(X)$  such that  $\mathbf{E}'_A = (1_X \times f)^*(\mathbf{E}_A) \otimes p_2^*(P)$ .

Let  $T$  be a subgroup of  $\mathrm{Eq}_0(\mathbf{D}(X), \mathbf{D}(X))$  induced by  $\mathrm{Aut}_H(X) \times \mathrm{Pic}^0(X)$ , where  $\mathrm{Aut}_H(X)$  is the group of isomorphisms preserving  $H$ . Then  $T$  preserves  $L$  and  $\Phi_{X \rightarrow X}^{\mathbf{E}_A} \bmod T$  is determined by  $A$ .

*Remark 1.10.* If  $X$  is an abelian surface with  $\mathrm{End}(X) \cong \mathbb{Z}$ , then  $\mathrm{NS}(X) = \mathbb{Z}H$  with  $(H^2) = 2D$  and for all autoequivalences  $\Phi$ ,  ${}^t\theta(\Phi)$  is of the form (1.5).

*Remark 1.11.* For an equivalence  $\Phi_{X \rightarrow X}^{\mathbf{E}} : \mathbf{D}(X) \rightarrow \mathbf{D}(X)$  such that  $\mathbf{E} \in \text{Coh}(X \times X)$  and  $\Phi_{X \rightarrow X}^{\mathbf{E}}$  preserves  $L$ . We set

$$(1.7) \quad B := {}^t\theta(\Phi_{X \rightarrow X}^{\mathbf{E}}) = \begin{pmatrix} p_1\sqrt{r_1} & p_2\sqrt{r_2} \\ q_1\sqrt{r_2} & q_2\sqrt{r_1} \end{pmatrix}$$

as in Theorem 1.6 (2). Then  $B^2$  is of the form (1.5). If  $(p_1 + q_2)p_2 > 0$  or  $p_2 = 0$ , then  $(\Phi_{X \rightarrow X}^{\mathbf{E}_{B^2}})^2 \equiv \Phi_{X \rightarrow X}^{\mathbf{E}_{B^2}}$  mod  $T$ . If  $(p_1 + q_2)p_2 < 0$  or  $p_1 + q_2 = 0$ , then  $(\Phi_{X \rightarrow X}^{\mathbf{E}_{B^2}})^2 \equiv \Phi_{X \rightarrow X}^{\mathbf{E}_{B^2}[-2]}$  mod  $T$ . Hence for the computation of  $h_t(\Phi_{X \rightarrow X}^{\mathbf{E}_{B^2}})$  is reduced to the computation of  $h_t(\Phi_{X \rightarrow X}^{\mathbf{E}_A})$ . In particular, if  $\text{NS}(X) = \mathbb{Z}H$ , then we can compute  $h_t$  for all equivalences.

Since the eigen equation of  $A$  in (1.5) is  $x^2 - (a + d)x + 1 = 0$ ,  $A$  has two real eigen value  $\alpha > \beta = 1/\alpha$  unless  $\text{tr } A = 0, 1, 2$ . If  $\text{tr } A = 0, 1$ , then  $A^4 = E, A^6 = E$ . If  $\text{tr } A = 2$ , then  $(A - E)^2 = 0$ , and hence

$$(1.8) \quad A^n = E + n(A - E).$$

Assume that  $\text{tr } A > 2$ . Then

$$(1.9) \quad \alpha, \beta \notin \mathbb{Q},$$

since  $\mathbb{Z}$  is integrally closed and  $0 < \beta < 1$ . We set

$$(1.10) \quad P := \frac{1}{\alpha - \beta}(A - \beta E), \quad Q := \frac{1}{\beta - \alpha}(A - \alpha E).$$

Then

$$(1.11) \quad A^n = \alpha^n P + \beta^n Q.$$

Let  $E$  be a semi-homogeneous sheaf on  $X$  with  $v(E) = (p^2, pqH, q^2D)$ ,  $p^2, pq, q^2D \in \mathbb{Z}$ . Then

$$(1.12) \quad \iota_X(v(L)) = u^t u, \quad u = \begin{pmatrix} p \\ q\sqrt{D} \end{pmatrix}.$$

We set

$$(1.13) \quad \begin{pmatrix} p_n \\ q_n\sqrt{D} \end{pmatrix} = A^n u = \alpha^n P u + \beta^n Q u.$$

Then

$$(1.14) \quad \iota_X(v(\Phi^n(E))) = \begin{pmatrix} p_n \\ q_n\sqrt{D} \end{pmatrix} \begin{pmatrix} p_n & q_n\sqrt{D} \end{pmatrix}.$$

Let

$$(1.15) \quad u_\alpha = \begin{pmatrix} 1 \\ s\sqrt{D} \end{pmatrix}$$

be an eigen vector with respect to  $\alpha$ . Thus  $s = \frac{\alpha - a}{bD}$ . By (1.9),  $(A - \beta E)u \neq 0$ , and hence

$$(1.16) \quad \lim_{n \rightarrow \infty} \frac{q_n}{p_n} = s.$$

The following is obvious.

**Lemma 1.12.** Assume that  $x^2 - \text{tr } Ax + \det A = (x - \alpha)(x - \beta)$ ,  $\alpha, \beta \in \mathbb{C}$ . Then the eigen equation of the representation matrix of the action of  $A$  on  $\text{Sym}_2(\mathbb{R}, D)$  is

$$(1.17) \quad (x - \alpha^2)(x - \alpha\beta)(x - \beta^2) = (x^2 - ((\text{tr } A)^2 - 2\det A)x + (\det A)^2)(x - \det A).$$

In particular, if  $|\alpha| \geq |\beta|$ , then  $|\alpha|^2$  is the spectral radius of this representation.

## 2. COMPUTATION OF ENTROPY

For an endofunctor  $\Phi : \mathbf{D}(X) \rightarrow \mathbf{D}(X)$ , let  $h_t(\Phi) : \mathbb{R} \rightarrow \{-\infty\} \cup \mathbb{R}$  be the entropy defined in [1, Defn. 2.5]. In this section, we shall compute the entropy for the equivalence  $\Phi := \Phi_{X \rightarrow X}^{\mathbf{E}_A}$  by using the following result.

**Proposition 2.1** ([1, Thm. 2.7] and [4, Prop. 3.8]). Let  $G, G'$  be split generators of  $\mathbf{D}(X)$  and  $F$  an endofunctor of  $\mathbf{D}(X)$  of Fourier-Mukai type such that  $F^n G, F^n G' \not\cong 0$  for all  $n > 0$ . Then

$$(2.1) \quad h_t(F) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \delta'_t(G, F^n G')$$

where

$$(2.2) \quad \delta'_t(M, N) := \sum_{m \in \mathbb{Z}} \dim \text{Hom}(M, N[m]) e^{-mt}, \quad M, N \in \mathbf{D}(X).$$

**Theorem 2.2.** Let  $\Phi := \Phi_{X \rightarrow X}^{\mathbf{E}^A}$  be an equivalence associated to  $A$  (see Definition 1.8).

(1) Assume  $b = 0$ . Then  $h_t(\Phi) = \log \rho(\Phi) = 0$ .

(2) Assume that  $b > 0$ . Then

$$(2.3) \quad h_t(\Phi) = \begin{cases} \log \rho(\Phi), & \text{tr } A \geq 2 \\ \log \rho(\Phi) - \frac{2}{3}t, & \text{tr } A = 1 \\ \log \rho(\Phi) - t, & \text{tr } A = 0. \end{cases}$$

(3) Assume that  $b < 0$ . Then

$$(2.4) \quad h_t(\Phi) = \begin{cases} \log \rho(\Phi) - 2t, & \text{tr } A \geq 2 \\ \log \rho(\Phi) - \frac{4}{3}t, & \text{tr } A = 1 \\ \log \rho(\Phi) - t, & \text{tr } A = 0. \end{cases}$$

*Remark 2.3.* If  $\text{tr } A \leq 2$ , then  $\rho(\Phi) = 1$ .

*Proof.* Let  $N$  be a line bundle with  $c_1(N) = mH$ ,  $m > 0$ . We take

$$(2.5) \quad G = \bigoplus_{-3 \leq i \leq -1} N^{\otimes i}, \quad G' = \bigoplus_{1 \leq i \leq 3} N^{\otimes i}$$

as split generators of  $\mathbf{D}(X)$  ([10]).

(1) Assume that  $b = 0$ . Then  $\Phi_{X \rightarrow X}^{\mathbf{E}^A}$  is an equivalence defined by  $\text{Aut}(X) \times \text{Pic}(X)$ . Then it is easy to see that

$$(2.6) \quad h_t(\Phi_{X \rightarrow X}^{\mathbf{E}^A}) = 0 = \log \rho(\Phi_{X \rightarrow X}^{\mathbf{E}^A}).$$

We next prove (2) and (3). So we assume that  $b \neq 0$ .

(I). We first treat the case where  $\text{tr } A > 2$ . We set

$$(2.7) \quad \begin{pmatrix} p_{i,n} \\ q_{i,n}\sqrt{D} \end{pmatrix} = A^n \begin{pmatrix} 1 \\ im\sqrt{D} \end{pmatrix}.$$

Then  $v(\Phi^n(N^{\otimes i})) = (p_{i,n}^2, p_{i,n}q_{i,n}H, q_{i,n}^2D)$ . For a sufficiently large  $n$ ,  $q_{i,n}/p_{i,n}$  is sufficiently close to  $s$  by (1.16). Hence we can take an integer  $m$  such that

$$(2.8) \quad \frac{q_{i,n}}{p_{i,n}} > -m$$

for all sufficiently large  $n$  (depending on  $m$ ). We note that  $\mu((\mathbf{E}_A^\vee)_{|X \times \{x\}}) = -\frac{a}{bD}H$  and

$$(2.9) \quad s + \frac{a}{bD} = \frac{\alpha - a}{bD} + \frac{a}{bD} = \frac{\alpha}{bD}.$$

Let  $E$  be a semi-homogeneous sheaf with  $\mu(E) = xH$ . We set  $\mu(\Phi^n(E)) = x_nH$ . Then

$$(2.10) \quad x_{n+1} = \frac{c + dx_n}{a + bDx_n}.$$

Assume that  $b > 0$ . Then (2.9) implies  $s > -\frac{a}{bD}$ . If  $|s - x| < \frac{\alpha-1}{|b|D}$ , then

$$(2.11) \quad \left| \frac{c + dx}{a + bDx} - s \right| \leq \frac{|x - s|}{\alpha}.$$

Hence  $x_n$  also satisfies the same condition. If  $E$  also satisfies  $x > -\frac{a}{bD}$ , then  $x_n > -\frac{a}{bD}$  for all  $n \geq 0$ . Hence  $\Phi^n(E) \in \text{Coh}(X)$  for  $n \geq 0$  by Proposition 1.3.

We set  $\Phi^n(N^{\otimes i}) = E_n^i[\phi_i(n)]$ ,  $E_n^i \in \text{Coh}(X)$  (cf. Proposition 1.2). For  $E := E_{n_0}^i$ ,  $\mu(E) = \frac{p_{i,n_0}}{q_{i,n_0}}H$ . Hence by (1.16), we get  $\Phi^{n'}(E_{n_0}^i) \in \text{Coh}(X)$  for  $n' \geq 0$  and  $n_0 \gg 0$ . Therefore  $\phi_i(n) \in 2\mathbb{Z}$  is constant for a sufficiently large  $n$ . Hence there are  $l_i$  such that  $\text{Hom}(G, \Phi^n(N^{\otimes i})[k]) = 0$  for  $k \neq l_i$  and

$$(2.12) \quad \begin{aligned} \delta'_t(G, \Phi^n(G')) &= \sum_{-3 \leq i \leq -1} \sum_{1 \leq j \leq 3} \chi(N^{\otimes i}, \Phi^n(N^{\otimes j}))e^{-l_j t} \\ &= \sum_{-3 \leq i \leq -1} \sum_{1 \leq j \leq 3} (\alpha^n a_{i,j} + \beta^n b_{i,j})^2 e^{-l_j t} \end{aligned}$$

where

$$(2.13) \quad \begin{aligned} \alpha^n a_{i,j} + \beta^n b_{i,j} &= \det \left( \begin{pmatrix} 1 \\ im\sqrt{D} \end{pmatrix}, (\alpha^n P + \beta^n Q) \begin{pmatrix} 1 \\ jm\sqrt{D} \end{pmatrix} \right) \\ &= \det \left( \begin{pmatrix} 1 \\ im\sqrt{D} \end{pmatrix}, A^n \begin{pmatrix} 1 \\ jm\sqrt{D} \end{pmatrix} \right) \end{aligned}$$

(see (1.11)). By (2.8),  $a_{i,j} \neq 0$  for all sufficiently large  $n$ . Hence  $\log \delta'_t(G, \Phi^n(G')) \sim 2n \log |\alpha|$  and

$$(2.14) \quad h_t(\Phi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \delta'_t(G, \Phi^n(G')) = 2 \log |\alpha| = \log \rho(\Phi).$$

Assume that  $b < 0$ . Then (2.9) implies  $s < -\frac{a}{bD}$ . If  $|s - x| < \frac{\alpha-1}{|b|D}$ , then

$$(2.15) \quad \left| \frac{c+dx}{a+bDx} - s \right| \leq \frac{|x-s|}{\alpha}.$$

Hence  $x_n$  also satisfies the same condition. If  $E$  also satisfies  $x < -\frac{a}{bD}$ , then  $x_n < -\frac{a}{bD}$ . Hence  $\Phi(E)[2] \in \text{Coh}(X)$ . We set  $(\Phi[2])^n(N^{\otimes i}) = E_n^i[\psi_i(n)]$ ,  $E_n^i \in \text{Coh}(X)$ . In this case,  $\psi_i(n)$  is constant for a sufficiently large  $n$ . Hence there are  $l_i$  such that  $\text{Hom}(G, \Phi^n(N^{\otimes i})[k]) = 0$  for  $k \neq l_i + 2n$  and  $\log \chi(G, \Phi^n(G')) \sim 2n \log |\alpha|$ . Hence

$$(2.16) \quad h_t(\Phi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \delta'_t(G, \Phi^n(G')) = 2 \log |\alpha| - 2t = \log \rho(\Phi) - 2t.$$

(II). Assume that  $\text{tr } A = 2$ . Then  $A^n = E + n(A - E)$ . We set  $s := \frac{1-a}{bD}$ . Let  $E$  be a semi-homogeneous sheaf with  $\mu(E) = xH$  and set  $\mu(\Phi^n(E)) = x_n H$ . Since

$$(2.17) \quad \begin{aligned} x_n &= \frac{nc + (n(d-1) + 1)x}{1 + n(a-1) + nbDx}, \\ x_n - s &= \frac{bDx + (a-1)}{(1 + n(a-1 + bDx))bD}, \end{aligned}$$

we have  $\lim_{n \rightarrow \infty} x_n = s$ . In the same way as in the case  $\text{tr } A > 2$ , we see that

$$(2.18) \quad h_t(\Phi) = \begin{cases} \log \rho(\Phi), & b > 0 \\ \log \rho(\Phi) - 2t, & b < 0. \end{cases}$$

(III). Assume that  $\text{tr } A = 1$ . Then  $A^3 = -E$ . It is easy to see that

$$(2.19) \quad (\Phi_{X \rightarrow X}^{\mathbf{E}_A})^2 \equiv \begin{cases} \Phi_{X \rightarrow X}^{\mathbf{E}_{A^2}} \mod T & b > 0 \\ \Phi_{X \rightarrow X}^{\mathbf{E}_{A^2}}[-2] \mod T & b < 0 \end{cases}$$

and

$$(2.20) \quad (\Phi_{X \rightarrow X}^{\mathbf{E}_A})^3 \equiv \begin{cases} [-2] \mod T & b > 0 \\ [-4] \mod T & b < 0 \end{cases}$$

(see Remark 1.9). Indeed  $\Phi_{X \rightarrow X}^{\mathbf{E}_A}((\mathbf{E}_A)_{|\{x\} \times X}) \in M_H(u^\vee)$  for  $b > 0$  and  $\Phi_{X \rightarrow X}^{\mathbf{E}_A}((\mathbf{E}_A)_{|\{x\} \times X})[2] \in M_H(u^\vee)$  for  $b < 0$ , where  $u := v((\mathbf{E}_A)_{|X \times \{x\} \times X}) = (b^2 D, baH, a^2)$ . Hence

$$(2.21) \quad h_t(\Phi_{X \rightarrow X}^{\mathbf{E}_A}) = \begin{cases} -\frac{2}{3}t & b > 0 \\ -\frac{4}{3}t & b < 0. \end{cases}$$

Assume that  $\text{tr } A = 0$ . Then  $(\Phi_{X \rightarrow X}^{\mathbf{E}_A})^2 \equiv [-2] \mod T$  implies

$$(2.22) \quad h_t(\Phi_{X \rightarrow X}^{\mathbf{E}_A}) = -t.$$

□

*Remark 2.4.* Our assumption of  $\Phi(L) = L$  is very strong in general. Indeed  $H$  is not preserved by the action of  $\text{Aut}(X)$  in general, and there is an abelian surface with an automorphism of positive entropy.

*Remark 2.5.* In [2], Ikeda introduced a mass growth  $h_{\sigma,t}$  of a stability condition  $\sigma$  and studied several properties. In our example,

$$(2.23) \quad h_{\sigma,t}(\Phi) = h_t(\Phi),$$

where  $\sigma$  be a stability condition such that  $Z_\sigma(E) := \langle e^{zH}, v(E) \rangle$  ( $z \in \mathbb{H}$ ) and  $\phi_\sigma(k_x) = 1$ :

We only explain the case where  $\text{tr } A > 2$ . We first note that  $\Phi$  preserves  $\sigma$ -stability for semi-homogeneous sheaves. We set

$$(2.24) \quad \alpha^n a_j(z) + \beta^n b_j(z) = \det \left( \begin{pmatrix} 1 \\ z\sqrt{D} \end{pmatrix}, (\alpha^n P + \beta^n Q) \begin{pmatrix} 1 \\ jm\sqrt{D} \end{pmatrix} \right).$$

Then by (1.11), we get  $Z_\sigma(\Phi^n(N^{\otimes j})) = (\alpha^n a_j(z) + \beta^n b_j(z))^2$ . Assume that  $b < 0$ . Then  $\phi_\sigma((\Phi[2])^n(N^{\otimes j}))$  is bounded. Hence by using [2, Thm. 1.1], we get  $h_{\sigma,t}(\Phi) = \log |\alpha|^2 - 2t = h_t(\Phi)$ .

**2.1. Gromov-Yomdin type conjecture by Kikuta and Takahashi.** Let  $X$  be an arbitrary abelian surface. We shall compute  $h_0(\Phi)$  and check the Kikuta and Takahashi's conjecture [4, Conjecture 5.3] for some endofunctors of  $\mathbf{D}(X)$ .

**Lemma 2.6.** *Let  $D$  be a divisor with  $(D^2) > 0$  and  $\eta \in \text{NS}(X)$ . There is no isotropic vector  $v \neq 0$  in  $(\mathbb{Q}e^\eta + \mathbb{Q}e^{\eta+D} + \mathbb{Q}e^{\eta+2D})^\perp$ .*

*Proof.* Replacing  $v$  by  $ve^{-\eta}$ , we may assume that  $\eta = 0$ . We set  $v = (r, \xi, a) \neq 0$ . Then  $(\xi^2) = 2ra$ . By the conditions, we get

$$(2.25) \quad a = 0, (D, \xi) - r \frac{(D^2)}{2} = 0, 2(D, \xi) - 2r(D^2) = 0.$$

Hence  $(D, \xi) = r(D^2) = 0$ . Since  $(D^2) > 0$ , we get  $r = a = (\xi^2) = 0$ . Since  $D^\perp$  is negative definite, we get  $\xi = 0$ . Therefore our claim holds.  $\square$

**Lemma 2.7.** *For a semi-homogeneous sheaf  $E$  and a line bundle  $L$ ,*

$$(2.26) \quad \sum_{k \in \mathbb{Z}} \dim \text{Hom}(L, E[k]) \leq \max\{4|\chi(L(pH), E)| \mid p = 0, \pm 1, \pm 2\},$$

where  $H$  is an ample divisor on  $X$ .

*Proof.* We first note that  $E$  is a semi-stable sheaf with respect to  $H$  by Proposition 1.1. Then  $E$  is  $S$ -equivalent to  $\oplus_i E_i$  such that  $E_i$  are stable with  $v(E_i) = v(E_j)$ . Hence it is sufficient to prove the claim for a stable semi-homogeneous sheaf  $E$ . We first assume that  $\chi(L, E) = 0$ . Then  $E$  is not 0-dimensional. We can easily show the following claim.

- (i) If  $(c_1(E \otimes L^\vee), H) > 0$ , then  $\dim \text{Hom}(L, E) = \dim \text{Hom}(L, E[1])$  and  $\text{Hom}(L, E[2]) = 0$ .
- (ii) If  $(c_1(E \otimes L^\vee), H) < 0$ , then  $\dim \text{Hom}(L, E) = 0$  and  $\dim \text{Hom}(L, E[1]) = \dim \text{Hom}(L, E[2])$ .
- (iii) If  $(c_1(E \otimes L^\vee), H) = 0$ , then  $\dim \text{Hom}(L, E) = \dim \text{Hom}(L, E[2]) \leq 1$ ,  $\dim \text{Hom}(L, E[1]) \leq 2$  and  $v(L) = v(E)$ .

For the case of (iii), obviously the claim holds by Lemma 2.6. So we shall treat the case of (i) and (ii). In these cases,  $E$  is not 0-dimensional. Replacing  $H$  by its translate, we can take an injective homomorphism  $E \rightarrow E(H)$ . Then we get

$$\dim \text{Hom}(L, E) \leq \dim \text{Hom}(L(pH), E)$$

for  $p \leq 0$ . We also see that

$$\dim \text{Hom}(E, L) \leq \dim \text{Hom}(E, L(pH))$$

for  $p \geq 0$ . By Lemma 2.6,  $\chi(L(pH), E) \neq 0$  for an integer  $p \in \{0, 1, 2\}$  and  $\chi(L(pH), E) \neq 0$  for an integer  $p \in \{0, -1, -2\}$ . Hence by using Proposition 1.3, we get

$$(2.27) \quad \begin{aligned} \dim \text{Hom}(L, E) &\leq \max\{0, \chi(L(pH), E) \mid p = -1, -2\} \\ \dim \text{Ext}^2(L, E) &\leq \max\{0, \chi(L(pH), E) \mid p = 1, 2\}. \end{aligned}$$

Thus

$$(2.28) \quad \sum_{k \in \mathbb{Z}} \dim \text{Hom}(L, E[k]) \leq \max\{2|\chi(L(pH), E)| \mid p = 0, \pm 1, \pm 2\}.$$

By Proposition 1.3, the same claim also holds if  $\chi(L, E) \neq 0$ .  $\square$

**Proposition 2.8.** *Let  $\Phi : \mathbf{D}(X) \rightarrow \mathbf{D}(X)$  be an endofunctor which is a composite of equivalences,  $f^*$  and  $f_*$ , where  $f : X \rightarrow X$  is a finite morphism. Then  $h_0(\Phi) \leq \log \rho(\Phi)$ .*

*Proof.* We use split generators  $G, G'$  in (2.5). For a finite morphism  $f : X \rightarrow X$ ,  $f^*$  and  $f_*$  send semi-homogeneous sheaves to semi-homogeneous sheaves. By Proposition 1.2, autoequivalences also send semi-homogeneous sheaves to semi-homogeneous sheaves up to shift. Hence  $\Phi^n(N^{\otimes j})$  is a semi-homogeneous sheaf up to shift. By Lemma 2.7, we have

$$(2.29) \quad \sum_{k \in \mathbb{Z}} \dim \text{Hom}(N^{\otimes i}, \Phi^n(N^{\otimes j})[k]) \leq \max\{4|\chi((N(pH))^{\otimes i}, \Phi^n(N^{\otimes j}))| \mid p = 0, \pm 1, \pm 2\}.$$

For any real number  $\lambda > \rho(\Phi)$ , we have

$$(2.30) \quad \lim_{n \rightarrow \infty} \frac{1}{\lambda^n} \Phi^n(N^{\otimes j}) = 0,$$

and hence

$$(2.31) \quad \lim_{n \rightarrow \infty} \frac{1}{\lambda^n} |\chi((N(pH))^{\otimes i}, \Phi^n(N^{\otimes j}))| = 0.$$



Then we have  $|\chi((N(pH))^{\otimes i}, \Phi^n(N^{\otimes j}))| \leq \lambda^n$  for sufficiently large  $n$ . Combining (2.29) with this estimate, we see that

$$(2.32) \quad h_0(\Phi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \delta'_0(G, \Phi^n(G')) \leq \log \lambda.$$

Since  $\lambda$  is arbitrary, we get  $h_0(\Phi) \leq \log \rho(\Phi)$ .  $\square$

*Remark 2.9.* Let  $X$  be a simple abelian variety of  $\dim X = d$ , that is, there is no subabelian variety  $Y$  of  $X$  with  $0 < \dim Y < d$ . Then for any line bundle  $L$ ,  $K(L) := \{x \in X \mid T_x^*(L) \cong L\}$  is a finite set, unless  $L \in \text{Pic}^0(X)$ . Hence if  $c_1(L) \neq 0$ , then  $(L^d) \neq 0$ . Then for a semi-homogeneous sheaf  $E$ , we see that  $\chi(E) \neq 0$  or  $\text{ch}(E) \in \mathbb{Z}_{>0} \text{ch}(\mathcal{O}_X)$ . Hence we also get  $h_0(\Phi) \leq \log \rho(\Phi)$ .

By [2, Thm. 1.2],  $h_0(\Phi) \geq \log \rho(\Phi)$ . Therefore we get the following result which support a Gromov-Yomdin type conjecture in [4, Conjecture 5.3].

**Proposition 2.10.** *Let  $X$  be an abelian surface and  $\Phi : \mathbf{D}(X) \rightarrow \mathbf{D}(X)$  an endofunctor in Proposition 2.8. Then  $h_0(\Phi) = \log \rho(\Phi)$ .*

### 3. AN EXAMPLE OF AUTOMORPHISM ON THE MODULI OF STABLE SHEAVES

Let  $M_H(v)$  be the moduli space of stable sheaves  $E$  on  $X$  with  $v(E) = v$  and  $K_H(v)$  be a fiber of the albanese map  $a : M_H(v) \rightarrow \text{Alb}(M_H(v)) = X \times \text{Pic}^0(X)$ .  $K_H(v)$  is an irreducible symplectic manifold which is derormation equivalent to a generalized Kummer manifold [13]. In [15, Prop. 3.50], we constructed an example of moduli space  $M_H(v)$  which have an automorphism of infinite order. Thus there is a Fourier-Mukai transform  $\Phi$  which induces an isomorphism  $g : M_H(v) \rightarrow M_H(v)$  such that  $g$  is infinite order. In this example, it is easy to see that a similar claim to [11] holds. Thus by using [8], we get

$$(3.1) \quad \frac{\dim K_H(v)}{2} h_0(\Phi) = h(g')$$

where  $g' : K_H(v) \rightarrow K_H(v)$  is an automorphism induced  $g$  by a Fourier-Mukai transform  $\Phi : \mathbf{D}(X) \rightarrow \mathbf{D}(X)$  in [15, Prop. 3.50]:

Let  $\mathcal{E}$  be a quasi-universal family on  $M_H(v) \times X$ . By  $T_x^*(\mathcal{E}) \otimes P_y$  ( $x \in X, P_y \in \text{Pic}^0(X)$ ), we have an isomorphism  $\psi : M_H(v) \rightarrow M_H(v)$  such that  $(\psi \times 1_X)^*(\mathcal{E}) \otimes L \cong T_x^*(\mathcal{E}) \otimes P_y \otimes L'$ , where  $L, L'$  are pull-backs of locally free sheaves on  $M_H(v)$ . Then

$$(3.2) \quad \begin{aligned} c_1(p_{M_H(v)!}(\text{ch}((\psi \times 1_X)^*(\mathcal{E}))\alpha^\vee)) &= c_1(p_{M_H(v)!}(\text{ch}(T_x^*(\mathcal{E}) \otimes P_y)\alpha^\vee)) \\ &= c_1(p_{M_H(v)!}(\text{ch}(\mathcal{E})T_{-x}^*(\alpha^\vee))) \\ &= c_1(p_{M_H(v)!}(\text{ch}(\mathcal{E})\alpha^\vee)) \end{aligned}$$

for  $\alpha \in v^\perp$ . Thus  $\psi^*(\theta_v(\alpha)) = \theta_v(\alpha)$ . Hence for an isomorphism  $g : M_H(v) \rightarrow M_H(v)$  induced by  $\Phi$ , we have an isomorphism  $g' : K_H(v) \rightarrow K_H(v)$  which induces the isomorphism  $\Phi : v^\perp \rightarrow v^\perp$ .

### 4. APPENDIX

Let  $(X, H)$  be a principally polarized abelian variety of  $\dim X = d$ . In [5], cohomological action of the group  $G$  of Fourier-Mukai transforms generated by  $\Phi_{X \rightarrow X}^{\mathbf{P}}$  and  $\otimes_{\mathcal{O}_X}(H)$  is described. In particular the action on the cohomology group generated by  $H$  is the action of  $\text{SL}(2, \mathbb{Z})$  on the  $d$ -th symmetric product of  $\mathbb{Q}^2$ . Then we can also compute  $h_t(\Phi_{X \rightarrow X}^{\mathbf{E}})$  of  $\Phi_{X \rightarrow X}^{\mathbf{E}} \in G$ , where  $\mathbf{E}$  is a coherent sheaf on  $X \times X$ . In particular  $h_t(\Phi_{X \rightarrow X}^{\mathbf{E}}) = \log |\alpha|^d - dt$  if  $\text{tr } A < -2$ .

*Remark 4.1.* For a semi-homogeneous sheaf  $E$  with  $\text{rk } E > 0$ , if  $c_1(E)$  is ample, then  $H^i(E) = 0$  for  $i \neq 0$  (cf. [6, Prop. 7.3]).

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